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On the connection between the stability of multidimensional positive systems and the stability of switched positive systems

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Abstract In this work, we study the connection of the stability of multidimensional positive systems with the stability of switched positive systems. In a previous work, we showed that the stability of a multidimensional positive system implies the stability of a related switched positive system. Here, we investigate the reciprocal implication.

1 Introduction

The study of stability conditions for switched positive systems has attracted the attention of several researchers (see, for instance, [4, 5, 8]). By relating a switched positive system with a multidimensional positive system, in [1] we provided a simple sufficient condition, that could be stated in terms of the spectral radius of a single matrix. However, it turns out that this sufficient condition is not necessary. In order to understand how far sufficiency is from necessity, here we search for additional conditions under which the stability of a switched positive system implies the stability of the related multidimensional positive system.

The remainder of this paper is organized as follows. In the next section, we make a brief introduction to multidimensional positive systems and their stability. The connection between the stability of these systems and the stability of switched pos-

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itive system is studied in Section 3. Finally, the paper ends with the conclusions in Section 4.

2 Multidimensional positive systems and their stability

The k-dimensional (kD) positive linear discrete systems of order n considered here are of the form

$$\Sigma^{kD}_{A_1,\dots,A_k}: \quad \omega(i) = \sum_{j=1}^{k} A_j \omega(i - e_j), \tag{1}$$

where $\omega(i) \in \mathbb{R}^n$ represents the non-negative local state at $i = (i_1,\dots,i_k) \in \mathbb{Z}^k$, $A_1,\dots,A_k \in \mathbb{R}^{n\times n}$ are non-negative matrices, $e_j \in \mathbb{Z}^k$ is the j-th unit vector and so $i - e_j = (i_1,\dots,i_{j-1}, i_j - 1, i_{j+1},\dots,i_k)$. Furthermore, letting $\bar{i} = \sum_{j=1}^{k} i_k$, the global state of $\Sigma^{kD}_{A_1,\dots,A_k}$ at time $\ell \in \mathbb{Z}_0^+$ is defined as the set of local states $\Omega_\ell = \{\omega(i) : \bar{i} = \ell\}$. Note that the notions of local and global state only coincide in the particular case of $k = 1$, when (1) describes a 1D system Σ_A such that $\omega(\ell) = A\omega(\ell - 1)$. Now, it is obvious that, given a non-negative initial state Ω_0, a sequence $\Omega_1, \Omega_2, \dots$ is uniquely determined by (1). The behavior of the global state sequences determines the stability properties of the system. In particular, $\Sigma^{kD}_{A_1,\dots,A_k}$ is said to be asymptotically stable if for every non-negative Ω_0 such that $||\Omega_0|| < \infty$, one has $\lim_{\ell\to+\infty} ||\Omega_\ell|| = 0$, where $||\Omega_\ell|| = \sup\{||\omega(i)||_2 : \bar{i} = \ell\}$ and $||\cdot||_2$ denotes the usual Euclidean norm. In the area of multidimensional systems, it is well known that the following condition (which does not explore the fact that the system is positive) is necessary and sufficient for the asymptotic stability of $\Sigma^{kD}_{A_1,\dots,A_k}$ [2]:

$$\det(I_n - \textstyle\sum_{j=1}^{k} z_j A_j) \neq 0 \ \forall (z_1,\dots,z_k) \in \mathbb{D}^k,$$

where $\mathbb{D}^k = \{(z_1,\dots,z_k) \in \mathbb{C}^k : |z_j| \leq 1,\ j = 1,\dots,k\}$ is the closed unit polydisc in $\mathbb{C}^k$. This condition is unpractical and is not in general easy to check. However, if we use the fact that the kD system is positive, then we get a simpler condition stated in the proposition below. The result was presented for $k = 2$ in [10]. We presented it for $k \geq 2$ in [1], but without a proof. We now prove it.

Proposition 1. *The kD positive system $\Sigma^{kD}_{A_1,\dots,A_k}$ is asymptotically stable if and only if the 1D positive system Σ_A with $A = A_1 + \dots + A_k$ is asymptotically stable.*

Proof. Let us assume that the kD positive system $\Sigma^{kD}_{A_1,\dots,A_k}$ is asymptotically stable. Suppose that the local states in Ω_0 are all equal to a non-negative $\omega_0 \in \mathbb{R}^n$, arbitrarily chosen. Then, it can be seen that the local states in Ω_ℓ are all equal to $(A_1 + \dots + A_k)^\ell \omega_0$ and hence that $||\Omega_\ell|| = ||(A_1 + \dots + A_k)^\ell \omega_0||_2$ for all $\ell \in \mathbb{Z}_0^+$. The asymptotic stability of the kD positive system implies that $\lim_{\ell\to+\infty} ||\Omega_\ell|| = 0$ and, therefore, $\lim_{\ell\to+\infty} ||(A_1 + \dots + A_k)^\ell \omega_0||_2 = 0$. Given that ω_0 is arbitrary, it follows that the 1D positive system Σ_A with $A = A_1 + \dots + A_k$ is asymptotically stable.

Now, let us assume that the 1D positive system Σ_A with $A = A_1 + \ldots + A_k$ is asymptotically stable. Suppose that the global state Ω_0 of the kD positive system $\Sigma^{kD}_{A_1,\ldots,A_k}$ is non-negative and such that $||\Omega_0|| < \infty$. Then, there exists $L \in \mathbb{R}^+$ such that, if $\omega(i)$ with $\bar{i} = 0$ is a local state in Ω_0, then $0_n \leq \omega(i) \leq L_n$, where 0_n and L_n are vectors of length n with all components equal to 0 and L, respectively, and where the inequalities should be understood component-wise. Now, let $\Psi : (\mathbb{Z}_0^+)^k \mapsto \mathbb{R}^{n\times n}$ be the map whose value $\Psi(i) = \Psi(i_1,\ldots,i_k)$ corresponds to the matrix resulting from the sum of all products in $\{A_1,\ldots,A_k\}$ where A_j appears i_j times for $j = 1,\ldots,k$, usually known as the Hurwitz product of $A_1,\ldots,A_k$ associated with i. For instance, if $k = 2$, then $\Psi(0,0) = I_n$, $\Psi(i_1,0) = A_1^{i_1}$ when $i_1 > 0$, $\Psi(0,i_2) = A_2^{i_2}$ when $i_2 > 0$ and $\Psi(i_1,i_2) = A_1\Psi(i_1-1,i_2) + A_2\Psi(i_1,i_2-1)$ when $i_1,i_2 > 0$ [3]. With this notation, if $\omega(i)$ with $\bar{i} = \ell$ is a local state in Ω_ℓ, we have

$$\begin{aligned}
||\omega(i)||_2 &= ||\textstyle\sum_{\bar{j}=\ell}\Psi(j)\omega(i-j)||_2 \\
&\leq ||\textstyle\sum_{\bar{j}=\ell}\Psi(j)L_n||_2 \\
&= ||(\textstyle\sum_{\bar{j}=\ell}\Psi(j))L_n||_2 \\
&= ||(A_1+\ldots+A_k)^\ell L_n||_2
\end{aligned}$$

and so $||\Omega_\ell|| \leq ||(A_1+\ldots+A_k)^\ell L_n||_2$ for all $\ell \in \mathbb{Z}_0^+$. The asymptotic stability of the 1D positive system Σ_A with $A = A_1 + \ldots + A_k$ implies that $\lim_{\ell\to+\infty} ||(A_1 + \ldots + A_k)^\ell L_n||_2 = 0$ and, therefore, $\lim_{\ell\to+\infty} ||\Omega_\ell|| = 0$. Finally, minding that Ω_0 is arbitrary, it follows that the kD positive system $\Sigma^{kD}_{A_1,\ldots,A_k}$ is asymptotically stable. $\square$

Remark 1. According to the proposition, checking the asymptotic stability of the kD positive system $\Sigma^{kD}_{A_1,\ldots,A_k}$ amounts to check the asymptotic stability of the 1D positive system Σ_A with $A = A_1+\ldots+A_k$, but this is very easy, because Σ_A is asymptotically stable if and only if the spectral radius of A is less than one, that is, $\rho(A) < 1$.

3 On the connection between the stability of multidimensional positive systems and the stability of switched positive systems

A switched positive linear discrete-time system of order n composed of k subsystems can be described by

$$\Sigma_{A_1,\ldots,A_k}: \quad x(\ell) = A_{\sigma(\ell-1)}x(\ell-1), \quad A_{\sigma(\ell-1)} \in \{A_1,\ldots,A_k\}, \tag{2}$$

where $x(\ell) \in \mathbb{R}^n$ represents the non-negative state vector at time $\ell \in \mathbb{Z}_0^+$, $A_1,\ldots,A_k \in \mathbb{R}^{n\times n}$ are non-negative matrices associated with the k subsystems and $\sigma : \mathbb{Z}_0^+ \mapsto \{1,\ldots,k\}$ is the switching signal. It is clear that, given a non-negative initial state

$$x(0) = x_0 \tag{3}$$

and a switching signal σ, a sequence $x(1), x(2), \ldots$ is uniquely determined by (2). The behavior of the state sequences determines the stability properties of the system. In particular, $\Sigma_{A_1,\ldots,A_k}$ is said to be uniformly asymptotically stable if it is uniformly stable (u.s.) and globally uniformly attractive (g.u.a.), *i.e.*:

- $\forall \varepsilon > 0$, $\exists \delta > 0$: $||x(0)||_2 < \delta \Rightarrow ||x(\ell)||_2 < \varepsilon \; \forall \ell \in \mathbb{Z}_0^+$, σ (u.s.);
- $\forall r, \varepsilon > 0$, $\exists \ell^\star \in \mathbb{Z}^+$: $||x(0)||_2 < r \Rightarrow ||x(\ell)||_2 < \varepsilon \; \forall \ell \geq \ell^\star$, σ (g.u.a.).

As is known, $\Sigma_{A_1,\ldots,A_k}$ is uniformly asymptotically stable if there exists a common quadratic Lyapunov function (CQLF) $V(x) = x^T P x$ such that

$$P \succ 0 \quad \wedge \quad P - A_j^T P A_j \succ 0 \quad j = 1, \ldots, k, \tag{4}$$

where T denotes transposition and $P \succ 0$ means that P is positive definite [9].

Now, consider the kD positive system $\Sigma^{kD}_{A_1,\ldots,A_k}$ described by (1) and whose global state $\Omega_0 = \{\omega(i) : \bar{i} = 0\}$ is determined by

$$\omega(0) = x_0, \quad \omega(i) = 0 \quad \bar{i} = 0 \wedge i \neq 0. \tag{5}$$

Note that, in $\Sigma_{A_1,\ldots,A_k}$, the state is updated in each step in a single direction, corresponding to the variable ℓ. Moreover, $\Sigma_{A_1,\ldots,A_k}$ has k operation modes, and when the j-th mode is active, the state update is made according to $x(\ell) = A_j x(\ell - 1)$. On the other hand, in $\Sigma^{kD}_{A_1,\ldots,A_k}$, the local state is updated in each step in k directions, corresponding to the variables $i_1, \ldots, i_k$ in i. In addition, the contribution of the j-th update direction to the overall update, given by

$$\begin{aligned}\omega(i_1, \ldots, i_j, \ldots, i_k) = \; & A_1 \omega(i_1 - 1, \ldots, i_j, \ldots, i_k) + \ldots + \\ & A_j \omega(i_1, \ldots, i_j - 1, \ldots, i_k) + \ldots + \\ & A_k \omega(i_1, \ldots, i_j, \ldots, i_k - 1),\end{aligned}$$

is represented by $A_j \omega(i_1, \ldots, i_j - 1, \ldots, i_k)$. Therefore, we can think of an update direction in $\Sigma^{kD}_{A_1,\ldots,A_k}$ as being associated with an operation mode in $\Sigma_{A_1,\ldots,A_k}$. Furthermore, it is easy to see that the local state $\omega(i) = \omega(i_1, \ldots, i_k)$ of $\Sigma^{kD}_{A_1,\ldots,A_k}$ equals the sum of all possibilities for the state $x(\ell)$ of the switching system $\Sigma_{A_1,\ldots,A_k}$ after $\ell = \bar{i}$ steps where the value of the switching signal is j for i_j times with $j = 1, \ldots, k$. Hence, the two systems have state evolutions that are closely related. This is illustrated in Figure 1 for $k = 2$. Note for instance that the value of $\omega(i) = \omega(i_1, i_2)$ along the i_j-axis evolves in the same manner as the value of $x(\ell)$ when the switching signal is such that $\sigma(\ell) = j$ for all ℓ. Also remark that the value of $\omega(1,1) = (A_1 A_2 + A_2 A_1) x_0$ results from the sum of the possible values for $x(2)$ after two steps where the value of the switching signal is 1 in one step and 2 in the other. Given the close relation between the state evolutions of both systems, it is not surprising that their stability properties are also related. This is clarified in the next proposition.

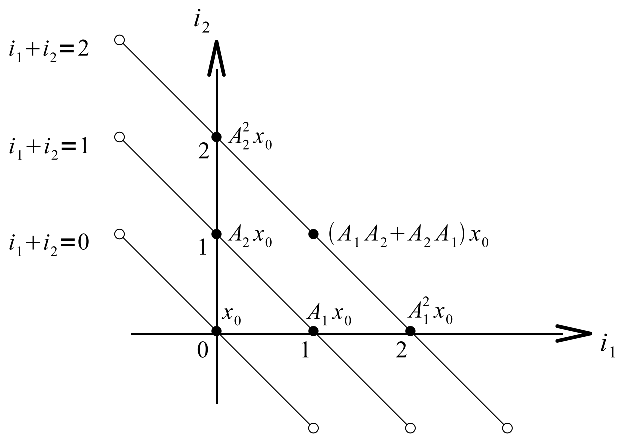

Fig. 1 State evolution of the 2D system $\Sigma^{2D}_{A_1,A_2}$ associated with the switching system Σ_{A_1,A_2}.

Proposition 2. *The switched positive system $\Sigma_{A_1,\dots,A_k}$ described by* (2),(3) *is uniformly asymptotically stable if the associated kD positive system $\Sigma^{kD}_{A_1,\dots,A_k}$ described by* (1),(5) *is asymptotically stable.*

We presented this result in [1]. In the following, we study the reciprocal implication and identify conditions under which the uniform asymptotic stability of the switched positive system $\Sigma_{A_1,\dots,A_k}$ implies the asymptotic stability of the associated kD positive system $\Sigma^{kD}_{A_1,\dots,A_k}$.

Start by noting that, as explained in Remark 1, a kD positive system $\Sigma^{kD}_{A_1,\dots,A_k}$ is asymptotically stable if and only if $\rho(A_1+\dots+A_k)<1$. In [1], we showed that, if $\rho(A_1+\dots+A_k)<1$, then it is possible to find a CQLF for the switched positive system $\Sigma_{A_1,\dots,A_k}$. Unfortunately, the converse is not true, as shown in the next example.

Example 1. Consider the switched positive system Σ_{A_1,A_2} described by (2),(3) with $k=2$ and

$$A_1=\begin{pmatrix}0.7 & 0\\ 0 & 0.1\end{pmatrix} \qquad A_2=\begin{pmatrix}0.4 & 0\\ 0 & 0.1\end{pmatrix}.$$

It is obvious that A_1 and A_2 are such that $\rho(A_1),\rho(A_2)<1$ and commute. Therefore, it is possible to find a CQLF for Σ_{A_1,A_2} [7]. Moreover, it can be seen that $\rho(A_1+A_2)=1.1 \not< 1$.

At this point, a natural question arises: is there a relation between the existence of a CQLF for a switched positive system $\Sigma_{A_1,\dots,A_k}$ and the value of $\rho(A_1+\dots+A_k)$? If the CQLF has no special form, then the answer is given by the following:

Proposition 3. *If the switched positive system $\Sigma_{A_1,\ldots,A_k}$ described by* (2),(3) *has a CQLF, then $\rho(A_1+\ldots+A_k)<k$.*

Proof. Let us assume that $V(x)=x^T Px$ is a CQLF for $\Sigma_{A_1,\ldots,A_k}$ such that $P\succ 0$ and

$$\begin{aligned} P-A_1^T PA_1 &\succ 0 \\ &\vdots \\ P-A_k^T PA_k &\succ 0. \end{aligned}$$

Then,

$$\begin{aligned} \left(P-A_1^T PA_1\right)+\ldots+\left(P-A_k^T PA_k\right) \succ 0 \Leftrightarrow \\ kP-\sum_{j=1}^{k} A_j^T PA_j \succ 0 \Leftrightarrow \\ k^2\left(kP^{-1}\right)^{-1}-\sum_{j=1}^{k} A_j^T PA_j \succ 0 \Leftrightarrow \\ \left(kP^{-1}\right)^{-1}-\sum_{j=1}^{k}\left(\frac{1}{k}A_j\right)^T P\left(\frac{1}{k}A_j\right) \succ 0. \end{aligned}$$

According to [6], the latter condition implies that the kD positive system $\Sigma^{kD}_{\frac{1}{k}A_1,\ldots,\frac{1}{k}A_k}$ is asymptotically stable. This in turn implies that $\rho(\frac{1}{k}A_1+\ldots+\frac{1}{k}A_k)<1$ and so $\rho(A_1+\ldots+A_k)<k$. $\square$

In the proposition just presented, no special form was assumed for the CQLF. However, if the CQLF for the switched positive system $\Sigma_{A_1,\ldots,A_k}$ is of a certain type, then the bound on $\rho(A_1+\ldots+A_k)$ can be tightened. This is clarified in the next result, which is the main contribution of this paper. It identifies conditions under which the uniform asymptotic stability of the switched positive system $\Sigma_{A_1,\ldots,A_k}$ implies the asymptotic stability of the associated kD positive system $\Sigma^{kD}_{A_1,\ldots,A_k}$. The proof is omitted because it is based on arguments similar to those previously used.

Proposition 4. *If the switched positive system $\Sigma_{A_1,\ldots,A_k}$ described by* (2),(3) *is uniformly asymptotically stable and has a CQLF $V(x)=x^T Px$ such that $P\succ 0$ and*

$$\begin{aligned} \frac{1}{k^2}P-A_1^T PA_1 &\succ 0 \\ &\vdots \\ \frac{1}{k^2}P-A_k^T PA_k &\succ 0, \end{aligned}$$

then $\rho(A_1+\ldots+A_k)<1$ *and the associated kD positive system* $\Sigma^{kD}_{A_1,\ldots,A_k}$ *described by* (1),(5) *is asymptotically stable.*

Remark 2. It is easy to see that a matrix P in the conditions above also satisfies $P-A_j^T PA_j \succ 0$ for $j=1,\ldots,k$. This means that in the previous proposition we are indeed asking for the existence of a CQLF for $\Sigma_{A_1,\ldots,A_k}$ of a special form.

The next example illustrates the application of Proposition 4.

Example 2. Consider the switched positive system $\Sigma_{A_1,\ldots,A_k}$ described by (2),(3) with non-negative diagonal matrices

$$A_j = \mathrm{diag}\left(\alpha_{j1},\ldots,\alpha_{jn}\right) \qquad j=1,\ldots,k.$$

Assume that $\Sigma_{A_1,\ldots,A_k}$ is uniformly asymptotically stable. Given that $\rho(A_1),\ldots,\rho(A_k)<1$, since the system is stable only if each subsystem is stable, and $A_1,\ldots,A_k$ commute, $\Sigma_{A_1,\ldots,A_k}$ has a CQLF $V(x)=x^T Px$ with P of diagonal form [7]:

$$P = \mathrm{diag}\left(p_1,\ldots,p_n\right),$$

where $p_1,\ldots,p_n>0$. Assume that $\frac{1}{k^2}P-A_j^T PA_j \succ 0$ for $j=1,\ldots,k$, that is, that the CQLF is in the conditions of the previous proposition. Then,

$$\begin{aligned} \frac{1}{k^2}P-A_j^T PA_j \succ 0 &\Leftrightarrow \\ \mathrm{diag}\left(p_1\left(\frac{1}{k^2}-\alpha_{j1}^2\right),\ldots,p_n\left(\frac{1}{k^2}-\alpha_{jn}^2\right)\right) \succ 0 &\Leftrightarrow \\ 0\le \alpha_{j1},\ldots,\alpha_{jn}<\frac{1}{k}& \end{aligned}$$

for $j=1,\ldots,k$. It is now simple to check that $\rho(A_1+\ldots+A_k)<1$ and hence the associated kD positive system $\Sigma^{kD}_{A_1,\ldots,A_k}$ described by (1),(5) is asymptotically stable.

4 Conclusions

In this paper we studied the relation between the stability of multidimensional positive systems and the stability of switched positive systems. Motivated by the fact that the stability of the former implies the stability of the latter [1], but not vice-versa, we searched for additional conditions under which the stability of a switched positive system implies the stability of a related multidimensional positive system. As a preliminary result, we showed that if the switched positive system has a common quadratic Lyapunov function of a certain type, then the associated multidimensional positive system is stable. In our opinion, this might be a step forward to obtain necessary and sufficient conditions for the stability of a new class of switched positive systems.

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